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ON THE MOTION OF GROUND WATERS IN A STRATUM BETWEEN TWO SLIGHTLY PERMEABLE STRATA UNDER CONDITIONS OF PRESSURE-NO-PRESSURE HEAD

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The problem of motion of ground waters in a stratum under conditions of pressure-no-pressure and seepage through the upper and lower strata is considered. This problem reduces to solving a system of functional equations whose solution can be obtained by the method of successive approximations. The behavior of the unknown boundary $x = x_1(t)$ which separates regions of pressure and no-pressure motion is investigated. Proof is given of the uniqueness of the solution around $t = t_0$.

Let us consider the problem of finding a differentiable function $x_1(t)$ such that $x_1(t_0) = 0$ and $x_1(t) > 0$ for $t \in [t_0, T]$, and the solution $u_i(x, t)$ (with continuous derivative u_{ix}) of equation

$$\frac{\partial u_i}{\partial t} = a_i \frac{\partial^2 u_i}{\partial x^2} - b_i(u_i - u_{0i}) \tag{1}$$

in region

$$\Omega_1 = \{(t, x):\ 0 < x < x_1(t),\ t_0 < t \leqslant T\} \quad \text{for } i = 1$$

$$\Omega_2 = \{(t, x):\ x_1(t) < x < \infty,\ t_0 < t \leqslant T\} \quad \text{for } i = 2$$

which satisfies conditions

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=+0} = q(t), \quad u_2|_{t=t_0+0} = \varphi(x), \quad u_2|_{x\to+\infty} = \text{const} \tag{2}$$

$$u_1|_{x=x_1(t)-0} = u_0^-, \quad u_2|_{x=x_1(t)+0} = u_0^+, \quad \left.\frac{\partial u_1}{\partial x}\right|_{x=x_1(t)-0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=x_1(t)+0} \tag{3}$$

where ($q'(t) \geqslant 0$ for $t > t_0$, with q' and $\varphi(x)$ being, respectively, a bounded and a reasonably smooth functions, $\varphi' \geqslant 0$ and $u_0^\pm = \text{const}$).

If $u_0^- \neq u_0^+$, solution $u_2(x, t)$ is continuous in region $\bar{\Omega}_2 \setminus (0,0)$.

This is the problem to which the investigation of pressure-no-pressure motion of ground waters reduces. For one-dimensional pressure motion in a stratum between two permeable strata the pressure head $H(x, t)$ of the basic level satisfies the equation [1]

$$\frac{\partial H}{\partial t} = a_2 \frac{\partial^2 H}{\partial x^2} - b_2 (H - H_0), \quad b_2 = \frac{1}{\mu_2}\left(\frac{k_{01}}{m_{01}} + \frac{k_{02}}{m_{02}}\right) \tag{4}$$

where H_0 denotes the constant and equal pressure heads of the upper and lower strata. The solution of this equation in region $\Omega = \{(t, x): 0 < x < \infty, 0 < t \leqslant t_0\}$ with conditions

$$H(x, 0) = H(x, t)_{x \to +\infty} = H_0, \quad \left.\frac{\partial H}{\partial x}\right|_{x=+0} = q(t), \ t > 0$$

is of the form

$$\Phi \equiv H - H_0 = -a_2 \int_0^t \frac{q(\tau)}{\sqrt{\pi a_2 (t-\tau)}} \exp\left[-\frac{x^2}{4a_2(t-\tau)} - b_2(t-\tau)\right] d\tau \tag{5}$$

In (4) and (5) a_2 and μ_2 are the coefficients of pressure conduction and of elastic recoil of the stratum [2], and k_{0i} and m_{0i} are the coefficients of filtration and stratification ($i = 1, 2$), respectively.

Let $q' \geqslant 0$, then the pressure head decreases in the course of time and, generally, at some instant of time $t = t_0 > 0$ attains at cross section $x = 0$ the value $H(0, t_0 - 0) = m_0$ (m_0 is the thickness of the basic stratum). If pumping is continued regions of motion under no-pressure (Ω_1) and pressure (Ω_2) appear in the stratum with a moving boundary $x = x_1(t)$ between them. The ground water level $h(x, t)$ in region Ω_1 then satisfies the equation

$$\mu_1 \frac{\partial h}{\partial t} = k \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) - \frac{k_{01}}{m_{01}}(h - H_0) + \varepsilon \tag{6}$$

Setting in this equation $h = \sqrt{2m_0 u_1}$ and linearizing it by the second method [1], we obtain Eq. (1), for $i = 1$, where

$$a_1 = \frac{kh_0}{\mu_1}, \quad b_1 = \frac{2k_{01}h_0}{\mu_1 m_{01}(h_0 + h_{01})}, \quad h_{01} = H_0 + \frac{\varepsilon m_{01}}{k_{01}}.$$

here a_1 and μ_1 are coefficients of level conduction and water release rate, h_0 is some mean value of $h(x, t)$, and ε is the rate of seepage from the upper stratum [3]. The pressure head $H(x, t)$ must for $t > t_0$ also satisfy Eq. (4) which assumes the form (1), when, $u_2 = H_2$ and $u_{02} = H_0$ are substituted in it.

Initial and boundary conditions are of the form (2), where $\varphi(x)$ is the same as function $\Phi(x, t_0)$. The first two conditions of (3) determine the level of ground waters and the pressure head along curve $x = x_1(t)$. When air is present above the ground water level, $2u_0^- = u_0^+ = m_0$, while in the absence of air $2u_0^- = m_0$ and $u_0^+ < m_0$ [3]. A particular case of a similar problem was investigated in [3, 4] by other methods.

For the solution of problem (1)–(3) we have the following integral representation:

$$u_1 = -a_1 \int_{t_0}^t G_1(x, t; 0, \tau) q(\tau) d\tau + a_1 \int_{t_0}^t G_1(x, t; x_1(\tau), \tau) \nu(\tau) d\tau + \tag{7}$$

$$\frac{b_1}{2}(u_0^- - u_{01}) \int_{t_0}^t \left[\operatorname{erf} \frac{x - x_1(\tau)}{2\sqrt{a_1(t-\tau)}} - \operatorname{erf} \frac{x + x_1(\tau)}{2\sqrt{a_1(t-\tau)}}\right] e^{-b_1(t-\tau)} d\tau + u_0^-$$

$$u_2 - H_0 = \Phi(x,t) + \frac{1}{2}[u_0^+ - H_0 - \Phi(x_1,t)] + \tag{8}$$
$$\frac{m_0 - u_0^+}{2} e^{-b_2(t-t_0)} \operatorname{erf} \frac{x}{2\sqrt{a_2(t-t_0)}} + \int_{t_0}^{t} \mu(\tau) e^{-b_2(t-\tau)} \times$$
$$\operatorname{erf} \frac{x - x_1(\tau)}{2\sqrt{a_2(t-\tau)}} d\tau - a_2 \int_{t_0}^{t} G_2(x,t;\ x_1(\tau),\tau)[\nu(\tau) - \Phi_x(x_1(\tau),\tau)] d\tau$$

where

$$G_i(x,t;\xi,\tau) = \frac{\exp[-b_i(t-\tau)]}{2\sqrt{\pi a_i(t-\tau)}} \left\{ \exp\left[-\frac{|x-\xi|^2}{4a_i(t-\tau)}\right] + \right.$$
$$\left. (2-i)\exp\left[-\frac{(x+\xi)^2}{4a_i(t-\tau)}\right] \right\}$$
$$\nu(t) = u_{1x}(x_1,t),\ \mu(t) = \frac{1}{2}\left\{ \frac{d\Phi}{dt}(x_1,t) + b_2[u_0^+ - H_0 - \Phi(x_1,t)] \right\}$$
$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\xi^2)\, d\xi \qquad (i = 1, 2)$$

With the use of the theorem about the normal derivative of the potential of a simple layer we obtain

$$\psi(t) \equiv \nu(t) - \Phi_x(x_1,t) = \frac{m_0 - u_0^+}{\sqrt{\pi a_2(t-t_0)}} \times \tag{9}$$
$$\exp\left[-\frac{x_1^2}{4a_2(t-t_0)} - b_2(t-t_0)\right] + 2\int_{t_0}^{t} G_2(x_1,t;x_1(\tau),\tau)\mu(\tau)d\tau -$$
$$2a_2 \int_{t_0}^{t} G_{2x}(x_1,t;\ x_1(\tau),\tau)\psi(\tau)\, d\tau$$

Then, by satisfying the first of conditions (3) and taking into consideration the continuity of u_{ix} (substituting in the first term in the right-hand part of (7) the identity $q(\tau) = q(\tau) - q_0 + q_0$, where $q_0 = q(t_0)$), from (7) we obtain

$$x_1(t) = 2\sqrt{\frac{a_1}{\pi}(t-t_0)} \exp\left[-\frac{x_1^2}{4a_1(t-t_0)}\right] + x_1 \operatorname{erf} \frac{x_1}{2\sqrt{a_1(t-t_0)}} + \tag{10}$$
$$\frac{a_1}{q_0}\int_{t_0}^{t} q_1(\tau) G_1(x_1,t;\ 0,\tau)\, d\tau - \frac{a_1}{q_0}\int_{t_0}^{t} G_1(x_1,t;x_1(\tau),\tau)\nu(\tau)\, d\tau -$$
$$b_1(u_0^- - u_{01}) \int_{t_0}^{t} \bar{G}_1(x_1,t;\ x_1(\tau),\tau)\, d\tau -$$
$$a_1 \int_0^{t-t_0} (1 - e^{-b_1\tau}) \exp\left[-\frac{x_1^2}{4a_1\tau}\right] \frac{d\tau}{\sqrt{\pi a_1 \tau}}$$

where

$$\bar{G}_1(x_1,t;\ \xi,\tau) = \frac{\exp[-b_1(t-\tau)]}{2\sqrt{\pi a_1(t-\tau)}} \left\{ \exp\left[-\frac{|x-\xi|^2}{4a_1(t-\tau)}\right] - \exp\left[-\frac{(x+\xi)^2}{4a_1(t-\tau)}\right] \right\}$$
$$q_1(t) = q(t) - q_0$$

When ψ and x_1 have been determined by means of functional equations (9) and (10), formulas (7) and (8) together with ψ and x_1 yield the solution of problem (1)–(3).

Let us consider the case of $u_0^+ = m_0$, when on previously made assumptions the system of functional equations (9) and (10) has a solution (the validity of this can be tested by the method of successive approximations).

Note that, if $\Phi\ (x,\ t)$ is specified by a formula other than (5), $d\Phi\ (x_1,t)\ /\ dt$ may have a singularity of the order of

$$O\,[(t-t_0)^{-1/2+\delta}],\quad \delta > 0$$

Thus, on the above assumptions the solution of problem (1)–(3) is derived by the method of successive approximations.

Let us investigate the behavior of curve $x = x_1\ (t)$ around $t = t_0$.

To estimate the right-hand part of (9) we make the substitution

$$\frac{d\Phi\,(x_1,\,t)}{dt} = x_1'(t)\,I(t) - q(0)\sqrt{\frac{a_2}{\pi t}}\exp\left(-\frac{x_1^2}{4a_2 t} - b_2 t\right) -$$

$$a_2\int_0^t \frac{q'(t-\tau)}{\sqrt{\pi a_2\tau}}\exp\left[-\frac{x_1^2(t)}{4a_2\tau} - b_2\tau\right]d\tau$$

$$I(t) = \int_0^t \frac{q(\tau)}{2\sqrt{\pi a_2(t-\tau)}}\,\frac{x_1(t)}{t-\tau}\exp\left[-\frac{x_1^2}{4a_2(t-\tau)} - b_2(t-\tau)\right]d\tau$$

and assume that

$$x_1\ (t) = c_1\ (t-t_0) + o\ (t-t_0),\quad c_1 = \text{const} \tag{11}$$

and that $\psi\ (t)$ is a continuous function.

We consider, for instance, the integral

$$J = \int_{t_0}^t G_2 x_1'(\tau)\,I(\tau)\,d\tau$$

Passing to variables

$$\tau = t_0 + \lambda\,(t-t_0),\qquad \frac{x_1(\tau)}{2\sqrt{a_2(t-\tau)}} = \lambda_1$$

we obtain

$$J = c_1 q_0\sqrt{\frac{t-t^0}{\pi a_2}} + o\,(\sqrt{t-t_0})$$

Moreover, owing to the continuity of ψ and the equality $\psi\ (t_0) = 0$ (which follows from the continuity of u_{2x} in Ω_2), we have

$$\int_{t_0}^{t_0} G_2\,(x_1(t),\,t;\ x_1(\tau),\,\tau)\,\psi d\tau = \frac{1}{2}\,\psi\,(t)\exp\frac{x_1}{2\sqrt{a_2(t-t_0)}} + o\,(\sqrt{t-t_0})$$

Estimating the remaining terms in the right-hand part of (9) by the method used for estimating J, we obtain

$$\psi\,(t) = \left[\frac{2c_1q_0}{\sqrt{\pi a_2}} - \frac{2q\,(0)}{\sqrt{\pi t_0}}\,e^{-b_2t_0} - c_0\right]\sqrt{t-t_0} + o\,(\sqrt{t-t_0}) \tag{12}$$

$$c_0 = \frac{2}{\sqrt{\pi}}\int_0^{t_0} q'(t_0-s)\,e^{-b_2 s}\,\frac{ds}{\sqrt{s}} \geqslant 0$$

Substituting (12) into (10) and computing related estimates, we have

$$x_1(t) = -{}^1/_2\,\psi(t)\sqrt{\pi a_1(t-t_0)} + o(t-t_0)$$

Hence, taking into account (11) and (12), we obtain

$$c_1 = \left(1+\sqrt{\frac{a_1}{a_2}}\right)^{-1}\left[\frac{q(0)}{q_0}\sqrt{\frac{a_1}{\pi t_0}}e^{-b_2 t_0} + \frac{c_0\sqrt{\pi a_1}}{2q_0}\right]$$

Consequently function $x_1(t)$ around $t = t_0$ is represented by a straight line (to within smalls of order $o(t - t_0)$) whose slope depends on the initial rate of flow from the stratum subjected to pressure, on the initial flow rate under conditions of pressure-no-pressure motion, on parameters of the stratum, and on the duration of motion under pressure.

On assumptions made with respect to $q(t)$ and $\varphi(x)$ the solution of problem (1)–(3) is unique, at least in the neighborhood of point $(0, t_0)$.

To prove this, let us assume that there exist two pairs of solutions: u_1, u_2 and x_1 along segment $[0, T_1]$ and v_1, v_2 and x_2 along segment $[0, T_2]$. Let

$$m(t) = \min[x_1(t), x_2(t)], \quad M(t) = \max[x_1(t), x_2(t)]$$
$$T = \min(T_1, T_2), \quad D_1 = \{(t, x): 0 < x < m(t), \quad t_0 < t \leqslant T\}$$
$$D_2 = \{(t, x): M(t) < x < \infty, \ t_0 < t \leqslant T\}$$

Let us consider function $w_2 = u_2 - v_2$ in region $\overline{D}_2$. Owing to

$$w_2(0, t_0) = 0, \qquad w_2|_{t\to t_0} = 0, \qquad w_2|_{x\to\infty} = 0$$

the point of maximum $P = \{M(t^\circ), t^\circ\}$ of function w_2 lies on curve $x = M(t)$. Hence $w_{2x}(P) < 0$ and $u_2(P) - v_2(P) > 0$. Noting that

$$u_2(x_1, t) = u_0^+, \quad v_2(x_2, t) = u_0^+, \quad u_{2x} > 0, \ v_{2x} > 0$$

we obtain $x_1(t^\circ) < x_2(t^\circ)$. On the other hand

$$w_1(x_1(t^\circ), t^\circ) = u_1(x_1(t^\circ), t^\circ) - v_1(x_1(t^\circ), t^\circ) = {}^1/_2 m_0 - v_1(x_1(t^\circ), t^\circ) > 0$$

consequently $w_{1x} > 0$. But for reasonably small $\delta_2 = (T - t_0)$ this contradicts the continuity of function w_x which in regions $\overline{D}_1$ and $\overline{D}_2$ is equal w_{1x} and w_{2x}, respectively. Hence in $\overline{D}_2$ function $u_2 \equiv v_2$ and $x_1 = x_2(t)$.

In fact, if we find a point $\tau \in [0, \delta_2]$ such that $x_2(\tau_0) > x_1(\tau_0)$, then $w_2(x_2(\tau_0), \tau_0) = u_2(x_2(\tau_0), \tau_0) - v_2(x_2(\tau_0), \tau_0) = u_2(x_2(\tau_0), \tau_0) - u_0^+ > 0$, which contradicts the identity $u_2 \equiv v_2$.

The uniqueness of u_1 follows from $x_1 = x_2$.

Note that in the proof of uniqueness no assumption was made as to the fulfilment of the equality $u_0^+ = m_0$.

We would point out that a certain class of problems with Stefan's condition (which is different from the second condition in (3)) along the unknown moving boundary were investigated in the monograph [6].

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ON AN APPROXIMATE METHOD OF SOLVING INTEGRAL EQUATIONS OF DYNAMIC CONTACT PROBLEMS

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Integral equations which originate in a number of contact problems concerned with the vibrations of stamps on the surface of domains, whose boundaries are at infinity (for example, on the surface of a layer, of a multilayer foundation, of a cylinder, a tube, etc.), are considered. Such problems reduce to integral equations of the first kind with a difference kernel containing oscillating members. The oscillations grow as the vibration frequency increases, and this either makes application of known methods of solving equations of the first kind difficult, or completely excludes such a possibility.

The possibility is studied of using a method of solving these equations and questions of its efficiency are discussed (*). In principle, the method permits construction of exact solutions of some equations approximating the initial equations, and errors in the approximate solutions are given.

1. The problem for an elastic layer of thickness h lying friction-free on a rigid foundation during vibrations of a stamp surface of width $2a$ adhering friction-free to its surface, results in an integral equation of the form

$$\int_{-a}^{a} k(x-t)\,q(t)\,dt = \pi f(x), \quad |x| \leqslant a \tag{1.1}$$

$$k(t) = \int_{\Gamma} K(u)\,e^{iut}du \tag{1.2}$$

$$K(u) = [u^2\sigma_2 \operatorname{cth}\sigma_2 - (u^2 - {}^1/_2\varkappa_2{}^2)^2\,\sigma_1{}^2 \operatorname{cth}\sigma_1]^{-1} \tag{1.3}$$

$$\sigma_k = \sqrt{u^2 - \varkappa_k{}^2}, \quad \varkappa_1{}^2 = \rho\omega^2h^2(2\mu+\lambda)^{-1}, \quad \varkappa_2{}^2 = \rho\omega^2h^2\mu^{-1}$$

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*) A.V.Belokon also expressed the possibility of using this method in one of the seminars of the elasticity theory department of Rostov State University.